REPESENTATION OF THE MOON'S PATH ALONG THE LINE OF THE EARTH'S ORBIT.

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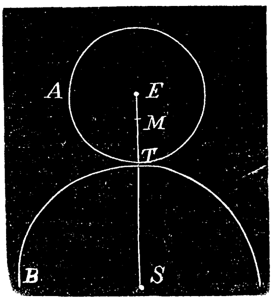
LET E represent the earth, S the sun, and ES the distance between them. On ES lay off ET so that ET : TS :: 1 : 13. Then is

$$\text{ET} : \text{ES} :: 1 : 14. \qquad (1)$$



With ET and TS as radii describe the circles TA and TB tangent to each other at T.

If we assume the moon's distance from the earth to be 240,000 miles and the earth's distance from the sun say 90,000,000 miles, these distances are as 1 to 375.

Let EM represent the moon's distance from the earth, then EM : ES :: 1 : 375,

and ET : ES :: 1 : 14, by (1); hence

EM : ET :: 14 : 375 :: 1 : 27, nearly.

Since circumferences are to each other as their radii, the circle AT will make 13 revolutions in rolling around the circle BT, and the point M will make 13 revolutions around the centre E during the same time.

If two such wheels are made and pointed pencils are fixd at E and M, by rolling AT toward the left around BT the pencil at M will describe the moon's path along the line of the earth's orbit as traced by the pencil at E.

A simpler plan is to cut from pasteboard any circle AT, and from centre E take EM $= \frac{1}{27}$ of the radius, insert pencils at E and M, and roll the circle along the inner edge of a black-board, and the relative path of the moon along the line of the earth's orbit will be traced.

PROB. 357 (p. 135, Vol. VIII).—"An elastic string without weight and of given length, has one end fixed in a perfectly smooth horizontal plane, and the other to a point in the surface of a sphere, the string being unwound. The sphere is projected on the plane from the fixed point with a linear velocity v and an angular velocity ω, winding the string on the circumference of a great circle; required the elongation of the string when fully stretched, and the subsequent motion of the sphere."

SOLUTION BY PROF. DE VOLSON WOOD, HOBOKEN, N. J.

Let r = the radius of the sphere, a = the original length of the string, ω = the initial angular velocity of the body, v = the initial velocity of the centre of the body, and t_1 = the time of winding the slack. Then

$$vt_1 + r\omega t_1 = a;$$

$$\therefore t_1 = \frac{a}{v + r\omega},$$

and the initial stretched part will be

$$vt_1 = \frac{va}{v + r\omega} = l \text{ (say)}.$$

Immediately following this time the string will be stretched, and the tension at first diminishes both the linear and angular velocities. Take the origin at the remote end of l for the variable motion. Let m = the mass of the body, s = the space passed over by the centre during time t, θ = the angular distance passed by the initial radius in the same time, k = the radius of gyration of the body, e = the coefficient of elasticity of the string, A = cross section of string, and λ = the elongation produced by the tension T of the string. Then Mariotte's law gives

$$T = \frac{eA\lambda}{l - r\theta}. \tag{1}$$

Assume that l is so long compared with $r\theta$ that the latter may be neglected, and let $B = eA \div l$, then

$$T = B\lambda.$$

The conditions of the problem give

$$d\lambda = ds + rd\theta; \tag{2}$$

$$\therefore \quad d^2\lambda = d^2s + rd^2\theta.$$

Also, for motion of the centre,

$$m\frac{d^2s}{dt^2} = -T = -B\lambda, \tag{3}$$

and for the rotary motion,

$$mk^2\frac{d^2\theta}{dt^2} = -Tr = -Br\lambda, \tag{4}$$

which two equations in the preceding give

$$\frac{d^2\lambda}{dt^2} = -\frac{B}{mk^2}\left(k^2 + r^2\right)\lambda = -D^2\lambda.$$

Integrating, observing that for $\lambda = 0$, $t = 0$, and $d\lambda \div dt = v + \omega$, we have

$$\lambda = \frac{v+\omega}{D}\sin Dt. \tag{5}$$

The elongation λ, will be a max. for sin $Dt = 1$, or $t = \pi \div 2D$, for which

$$\lambda = \frac{v+\omega}{D} = \frac{k\surd(ml)}{\surd[eA(k^2+r^2)]}(v+\omega).$$

The *time* of producing the maximum stretch of the string is independent of the initial motions. When the string returns to its original length, λ will again be zero, and sin $Dt = 0$, or $Dt = \pi$; $\therefore t = \pi \div D$.

All the circumstances of the variable motion may be determined by integrating equations (3) and (4). Integrating, after substituting from equation (5), observing that for $t = 0$, $ds \div dt = v$, $s = 0$, $d\theta \div dt = \omega$ and $\theta = 0$, we have, if we put F for $eA(v+\omega) \div mlD^2$,

$$\frac{ds}{dt} = F[\cos Dt - 1] + v, \tag{6}$$

$$s = \frac{F}{D}\Big[\sin Dt - Dt\Big] + vt, \tag{7}$$

$$\frac{d\theta}{dt} = F\frac{r}{k^2}\Big[\cos Dt - 1\Big] + \omega, \tag{8}$$

$$\theta = \frac{F}{D}\frac{r}{k^2}\Big[\sin Dt - Dt\Big] + \omega t. \tag{9}$$

For the maximum of (5), $d\lambda \div dt = 0$, which in (2) gives

$$\frac{ds}{dt} = -r\frac{d\theta}{dt},$$

which combined with (6) and (9) gives $\cos Dt - 1 = -1$; $\therefore Dt = \frac{1}{2}\pi$ as before found, and serves as a check upon the work. The relation $ds = -rd\theta$ shows that the direction of one of the motions changes signs. At the point where the linear motion is reversed $ds \div dt = 0$, and for this we have

$$t_2 = \frac{1}{D}\cos^{-1}\left(1 - \frac{v}{F}\right);$$

and if the direction of the rotation is reversed, $d\theta \div dt = 0$, and (8) gives

$$t_3 = \frac{1}{D}\cos^{-1}\left(1 - \frac{\omega k^2}{Fr}\right);$$

from which it appears that if $v < (k^2\omega \div r)$ the motion of the centre will be reversed, but otherwise the angular motion will be reversed. The value of t_2 in the former case will be less than $\pi \div 2D$. Both motions will change at the instant of greatest elongation if $rv = k^2\omega$.

If the values of t_2 and t_3 are both less than $\pi \div D$, one motion will change signs before the instant of greatest elongation and the other after; otherwise only one will change signs. To find the total variable movement, make $Dt = \pi$, and (7) and (9) give

$$s = \left(v - F\right)\frac{\pi}{D},$$

$$\theta = \left(\omega - F\frac{r}{k^2}\right)\frac{\pi}{D}.$$

If (6) reduces to zero when $Dt = \pi$, the body would be at rest at the moment the string regains its original length and $F = \frac{1}{2}v$, but it would still have an angular velocity of $\omega + (rv \div k^2)$ as shown by (8). Similarly, if the rotary motion is destroyed at that instant, the linear velocity will be $v + (k^2\omega \div r)$, and will continue uniform. It may be shown that the kinetic energy of the moving body at the end of the variable motion is the same as at the beginning.

I have not attempted to solve the general case represented by equation (1). It is evidently very intricate.

NOTE ON PROF. CASEY'S TREATMENT OF PROB. 361, BY PROF. E. W. HYDE. — Referring to Prof. Casey's figure on page 194, Vol. VIII, the curves *nxm* and *pyw* are *not* semi-ellipses. They are tortuous curves whose projections on the plane *ABC* are arcs of an *hyperbola*, those on a plane perpendicular to *BM* equal curves, forming together a curve of the 4th order, and those on a plane perpendicular to these two planes, the circle which is the right section of the cylinder.

The differential expression for the volume is easily written but can be integrated only by expansion into series. Thus the equation of the cone is $x^2+y^2 = (R^2 \div a^2)(z-a)^2$, and that of the cylinder, $y^2+z^2 = r^2$; whence

$$V = 8R\int_0^r\int_0^{\sqrt{(r^2-z^2)}} dz\,dy\sqrt{\left[\left(\frac{z-a}{a}\right)^2 - \left(\frac{y}{R}\right)^2\right]} = 4R\int_0^r dz\left\{\sqrt{(r^2-z^2)}\times\sqrt{\left[\left(\frac{z-a}{a}\right)^2 - \frac{r^2-z^2}{R^2}\right]} + \left(\frac{z-r}{a}\right)^2 R\sin^{-1}\frac{a}{R}\frac{(r^2-z^2)^{\frac{1}{2}}}{z-a}\right\}$$

[Mr. Eastwood puts for the equation of the cone and cylinder, respectively, $y^2+x^2 = z^2\tan^2\beta$, and $(z-c)^2+x^2 = r^2$, and gets

$$V = 4\int\sqrt{(r^2-x^2)}\left\{[\sqrt{(r^2-x^2)}+c]^2\tan^2\beta - x^2\right\}^{\frac{1}{2}}dx.$$

Mr. Heaton puts x = the distance of a horizontal plane through the cylinder from the axis of the cylinder, and finds, for the area of such plane,

$$A = 2\sqrt{\left\{(r^2-x^2)[(R^2\div a^2)(b+x)^2 - r^2 + x^2]\right\}} + 2(R^2\div a^2)(b+x)^2\times \sin^{-1}\sqrt{\left\{(r^2-x^2)\div[(R^2\div a^2)(b+x)^2]\right\}};$$ and therefore

$$V = 2\int_{-r}^{+r}\left\{\left[(r^2-x^2)\left(\frac{R^2}{a^2}(b+x)^2 - r^2 + x^2\right)\right]^{\frac{1}{2}} + \frac{R^2}{a^2}(b+x)^2\times \sin^{-1}\left(\frac{a\sqrt{(r^2-x^2)}}{R(b+x)}\right)\right\}dx.$$